Numerical Solutions of Fredholm Integral Equations using Collocation-Tau Method

Domingo, Augustine 'Dele

***Abstract:** Many problems arising in mathematics and in particular, applied mathematics or mathematical physics can be formulated in two but related ways, namely as differential or integral equation. Not all of such equations can be solved analytically; hence, numerical techniques are desirable. A tau collocation approach that combines the tau method with the idea of collocation for the solution of integral equations of Fredholm type is considered herein. The scope of the Lanczoz-Tau method is thus extended so that integral equations can also be solved numerically with the tau process. This work is supported with numerical evidences which show that the desired solution is accurately estimated by the resulting Tau approximant.*

***Keywords:** Collocation-Tau method, Fredholm Integral equations, Chebyshev polynomials, Linear Ordinary Differential equations.*

I. INTRODUCTION

Over the years, much emphasis has been placed on the solution of differential equations with less work being done on integral equations because they are a bit more difficult to solve than differential equations. Due to the difficulty, we often have to appeal to numerical methods for approximate solutions which include collocation-tau method considered in this work. Although few authors, the likes of Onumanyi and Taiwo [2,3], have only worked on singularly perturbed second order differential equation while Oladejo, Mojeed and Olurode [6] used the application of cubic spline to the solution of Fredholm integral equation. In this work, however, collocation-tau method is applied to the solution of Fredholm integral equation.

Therefore in this work, the general integral equation [1,4] used is of the form

$$y(x)+\lambda\int_a^b k(x,s)y(s)ds = g(x) \qquad 1$$

where

- $k(x,s)$ is a kernel
- $y(x)$ is an unknown function, $g(x)$ is a given function, and
- λ is scalar parameter, $y(s)$ is known (and often called the driven term)

The approximate solution $[1,4]$ used as defined as;

$$y(x)+\lambda\int_a^b k(x,s)\left[\sum_{r=0}^{N} a_r T_r(s)\right]ds = \sum_{r=0}^{N} a_r T_r(x) + H_n(x) \qquad 2$$

where

$$T_r(x) = Cos\left\{rCos^{-1}\left(\frac{(2x-a-b)}{(b+a)-1}\right)\right\}$$

$$= \sum_{k=0}^{r} C_k^{(r)} X^k$$

is the $r-th$ degree Chebyshev polynomial valid in the finite interval $[a,b]$

II. METHODOLOGY

Tau method: The lanczos Tau solution is implicitly defined by Linear Ordinary Differential Equation of order N with polynomial coefficients having a polynomial right hand side of the form;

$$Ly(x) = \sum_{i=0}^{N} Pi(x)y^{(i)}(x) = f(x);\ a \le x \le b \qquad 3$$

Where $y(x)$ satisfies the supplementary conditions

$$Ly(x_{ij}) = \sum_{i=0}^{N} a_{ij} y^{(i)}(x_{ij}) = \alpha_j; \quad j = 1(1)N \qquad 4$$

The constants $a_{ij}, x_{ij}, \alpha_j;\ i = 0(1)N-1, \quad j = 1(1)N$ are given real numbers where $X_{ij} \in [a,b]$ at which (3) are specified and

$$f(x) = \sum_{r=0}^{N} a_r T_r(x)$$

The basic idea of the Tau method, as conceived by Lanczos is the addition to (3) of a small perturbation term $H_N(x)$ which causes

$$Ly_n(x) = \sum_{i=0}^{N} Pi(x)y_n^{(i)}(x) = f(x) + H_N(x)$$

to have an exact polynomial solution of degree N of the

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form;

$$y_N(x) = \sum_{r=0}^{N} a_r T_r(x)$$

III. Integration procedure of the tau method

Consider the $m-th$ order Linear Ordinary Differential equation;

$$P_m(x)\frac{d^m y(m)}{dx^m} + \cdots + P_0(x)y(x) = f(x); \quad 0 \le x \le 1 \qquad 5$$

where $P_i(x); \quad i = 0(1)m$ are polynomials.

By a polynomial,

$$y_n(x) = a_0 + a_1 x + \cdots + a_n x^n; \quad n > 0 \qquad 6$$

we thereby approximate $y(x)$ by a polynomial

$$y_N(x) = \sum_{r=0}^{N} a_r T_r \qquad 7$$

and determine the coefficients of (6) such that $y_n(x)$ satisfies the equation (5) perturbed by a small term. That is $y_N(x)$ satisfies

$$\sum_{i=0}^{m} P_i(x) y_n^{(i)}(x) = f(x) + H_N(x) \qquad 8$$

where the perturbation $H_N(x)$ is of the form

$$H_N(x) = \tau T_{N+1}(x).$$

The integrated form of (5) is

$$I_D(y(x)) = \iint \cdots \int f(x) dx_1 dx_2 \cdots dx_{m-1} + C_m(x) \qquad 9$$

where $C_m(x)$ denotes an arbitrary polynomial of degree $(m-1)$ arising from the constants of integration. The Tau approximant $y_N(x)$ of (7) satisfies the perturbed equation,

$$y_N(x) = a_0 + a_1 x + \cdots + a_N x^N + H_N(x) \qquad 10$$

For the purpose of this discussion, Fredholm integral equation [1,5] of the second kind of this form is considered;

Standard Collocation Method: In this subsection, we shall consider the standard collocation method which requires an equal spacing of collocation points with specified range of problem at hand [ ]. If we have an integral equation of the form,

$$y(x) + \lambda \int_a^b k(x,s) y(s) ds = f(x) \qquad 11$$

The problem within the integrand is solved after which we collocate the entire equation at $(x_k = kh, k = 1,2,\cdots, N+1)$ step size

$$h = \frac{(b-a)}{N},$$

$$x_k = \frac{(b-a)k}{(N+1)}, \quad k = 1,2,\cdots, N+1$$

where N represents the degree of approximation

This will generate a system of linear equations (along with the recurrence relation and the end conditions), which will be solved by MATLAB 10.0.

Chebyshev Collocation Method: herein, equally spaced collocation method is considered. The equally spaced collocation method [7,8] is given by the formula

$$x_i = a + \frac{(b-a)i}{N+2}; \qquad i = 1,2,\cdots, N+2$$

This leads to a system of linear equations with a recurrence relation given in equation (2) which is then solved by MATLAB 10.0 to obtain the unknown constants which are then substituted in the approximate solution.

Orthogonal Collocation Method: There is a growing literature on spline collocation methods for numerical solution of initial value problems for Ordinary Differential Equation. This subsection makes use of the orthogonal collocation based on the zeros of Chebyshev polynomials which is given as,

$$x_i = \frac{1}{2}\left\{(b-a)Cos\left[\frac{(2i+1)\pi}{2n}\right] + (a+b)\right\}; \quad i = 0(1)N+1$$

This, like the other two techniques, generates a system of algebraic equations which is solved using MATLAB 10.0 for the unknowns which are substituted in the approximate

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solution.

These techniques are demonstrated on some numerical examples and the efficiency, accuracy, and amount of work are demonstrated.

Numerical Experiment: Error is defined in all cases as,

$$Error = Max\ |Exact - Approx.| = Max.|y(x) - y_N(x)| a \le X \le b$$

Example 1: Consider the integral equation

$$\int_0^1 xe^{xs} u(s)ds = e^x - u(x) \qquad 12$$

where $N = 4$ is the degree of approximation, equation (12) becomes

$$\int_0^1 xe^{xs} \left[\sum_{r=0}^{4} a_r \sum_{k=0}^{r} C_k^{(r)} S^k ds \right] = e^x - \sum_{r=0}^{4} a_r \sum_{k=0}^{r} C_k^{(r)} X^k + \tau T_5(x) \qquad 13$$

The results of the errors obtained from comparing the solutions of the three collocation methods with the exact solutions are tabulated below.

Example 2: Consider the integral equation

$$\int_0^1 (x + s)u(s)ds = u(x) - \frac{3}{2}x + \frac{5}{6} \qquad 14$$

where $N = 3$ is the degree of approximation, equation (14) becomes

$$\int_0^1 (x + s)\left[\sum_{r=0}^{3} a_r \sum_{k=0}^{r} C_k^{(r)} S^k \right] ds = \sum_{r=0}^{3} a_r \sum_{k=0}^{r} C_k^{(r)} X^r - \frac{3}{2}x + \frac{5}{6} + \tau T_4(x)$$

The results of the errors obtained from comparing the solutions of the three collocation methods with the exact solutions are tabulated below.

Example 3: Consider the integral equation

$$\int_0^1 e^{st} x(t)dt = \frac{(e^{s+1} - 1)}{s + 1} \qquad 15$$

where $N = 5$ is the degree of approximation, equation 15 is becomes

$$\int_0^1 e^{st} \left[\sum_{r=0}^{5} a_r \sum_{k=0}^{r} C_k^{(r)} t^k \right] dt = \frac{(e^{s+1} - 1)}{s + 1} + \tau T_6(s)$$

The results of the errors obtained from comparing the solutions of the three collocation methods with the exact solutions are tabulated below.

IV. RESULTS AND DISCUSSION

The results of the three collocation points (Standard Collocation Method, Orthogonal Collocation Method, Orthogonal Collocation Method) used in this paper are compared with the exact solution.

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Table 1: Comparison of Solution Parameters obtained for Example 1

Coefficients	*Method* 1	*Method* 2
a_0	0.999999999	0.999999999
a_1	$6.227436982\times10^{-10}$	5.994090061×10^{-9}
a_2	$-4.570028802\times10^{-10}$	$-2.957453789\times10^{-9}$
a_3	3.63953683×10^{-10}	-9.5528047×10^{-10}
a_4	$1.902982382\times10^{-10}$	0
τ	$3.1410859907\times10^{-11}$	0

Table 2: Comparison of Solution Parameters obtained for Example 2

Coefficients	*Method* 1	*Method* 2	*Method* 3
a_0	-0.500000004	-0.500000001	-0.5000000008
a_1	0.50000000018	0.5	0.499999998
a_2	$-3.026009509\times10^{-9}$	0	0.000000002
a_3	2.277904561×10^{-9}	0	3.33333335×10^{-10}
τ	$-1.423227062\times10^{-4}$	0	1.00000001×10^{-19}

Table 3: Comparison of solution Parameters obtained for Example 2

Coefficients	*Method* 1	*Method* 2
a_0	1.753312014	1.753307199
a_1	0.850256605	0.849557206
a_2	0.105053265	0.105075473
a_3	8.511503871×10^{-3}	7.355222218×10^{-3}
a_4	1.838917985×10^{-4}	2.404207755×10^{-8}
a_5	8.289650465×10^{-7}	0
τ	1.15321117×10^{-8}	0

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Table 4: Comparison of Solutions obtained for Example 1 at Selected Nodes

x	*Exact Solution* at $x=1$	*Approximate Solution with method* 1	*Approximate Solution with method* 2
0.1	1	0.999999999	0.999999993
0.2	1	0.999999999	0.999999995
0.3	1	0.999999999	0.999999997
0.4	1	0.999999999	1.000000000
0.5	1	0.999999999	1.000000002
0.6	1	0.999999999	1.000000003
0.7	1	0.999999999	1.000000004
0.8	1	0.999999999	1.000000004
0.9	1	0.999999999	1.000000003
1	1	0.999999999	1.000000001

Table 5: Comparison of Solutions obtained for Example 2 at Selected Nodes

3	*Exact Solution* at $x=1$	*Approximate Solution with method* 1	*Approximate Solution with method* 2	*Approximate Solution with method* 3
0.1	-0.9	-0.900000005	-0.900000001	-0.900000005
0.2	-0.8	-0.800000002	-0.800000001	-0.800000007
0.3	-0.7	-0.7	-0.700000001	-0.700000008
0.4	-0.6	-0.6	-0.600000001	-0.600000009
0.5	-0.5	-0.500000001	-0.500000001	-0.500000001
0.6	-0.4	-0.400000002	-0.400000001	-0.400000001
0.7	-0.3	-0.300000003	-0.300000001	-0.300000001
0.8	-0.2	-0.200000004	-0.200000001	-0.200000001
0.9	-0.1	-0.100000004	-0.100000001	-0.100000009
1	0.0	$-2.748104948\times10^{-9}$	-0.000000001	$-7.666666667\times10^{-19}$

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Table 6: Comparison of Solutions obtained for Example 3 at Selected Nodes

s	Exact Solution at e^s	Approximate Solution with method 1	Approximate Solution with method 2
0.1	1.105170918	1.105363463	1.105671585
0.2	1.221402758	1.22155491	1.22036211
0.3	1.349858808	1.34979345	1.348976323
0.4	1.491824698	1.491572923	1.490904106
0.5	1.648721271	1.648442641	1.64823175
0.6	1.8221188	1.822007898	1.822371455
0.7	2.013752708	2.01393048	2.014735428
0.8	2.225540928	2.225929175	2.226735882
0.9	2.45960311	2.459780279	2.459785038
1	2.718281828	2.717318109	2.715295124

It is seen from the tables shown above that the solutions obtained are almost the same. However, in terms of accuracy, efficiency, and computational cost, Chebyshev method gave the best results among the three proposed methods because of its closeness to exact solution.

V. Conclusion

Fredholm equations of the second kind have been considered in this work for solution with a collocation-tau method. Equations of the third kind can also be considered in the light of the methods herein presented which are recommended for further investigation.

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